# 14. Double Integrals Over a General Region Part 1

In the previous section we discussed double integrals over a rectangle $R$.

In this section, we will talk about:

- Fubini's Theorem
- Double Integrals Over a General Region
    - Regions of Type 1, 2, and 3.
    - Examples of computing the double integrals

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Fubini's theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

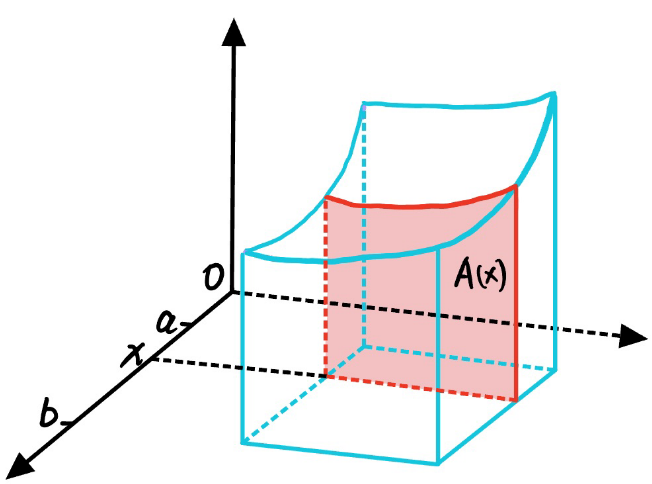
**Fubini's Theorem**

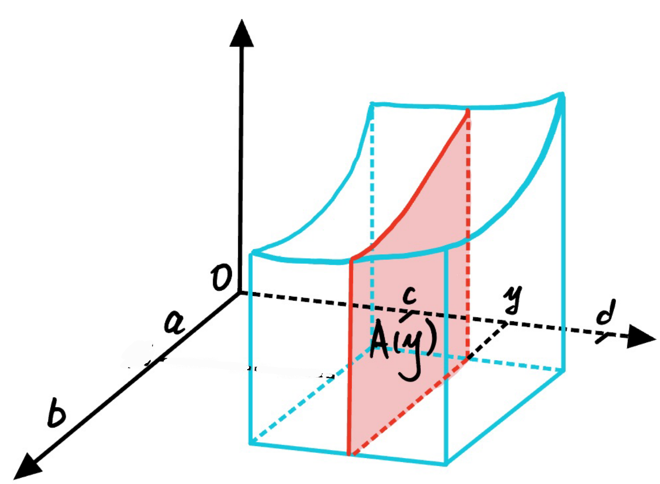
If $f$ is continuous on the rectangle $R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, then

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

More generally, this is true if we assume that $f$ is bounded on $R$, $f$ is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

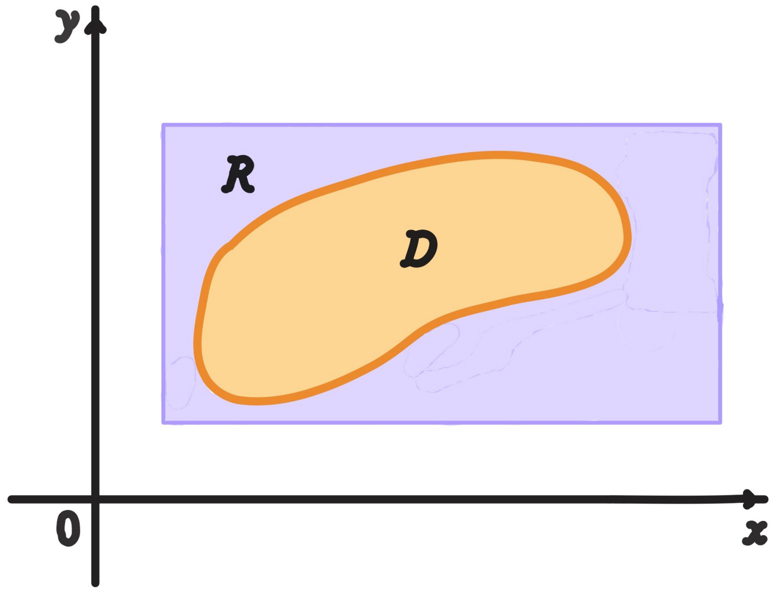
Intuitively, if $f(x, y) \geq 0$, this means we can take a "slice" of the solid in two different ways to compute the volume:



In this section, we consider defining the double integral of a real-valued function $f(x, y)$ over more general regions in $\mathbb{R}^2$.

Suppose $f(x, y)$ is defined on a bounded region $D$, which means $D$ can be inside of a rectangular region $R$ indicated as the following:




We define a new function $F(x, y)$ with domain $R$ by

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases} \tag{1}$$

If the double integral of $F$ exists over $R$, then we define the double integral of $f$ over $D$ by

$$\iint_D f(x,y)dA = \iint_R F(x,y)dA \quad \text{where } F \text{ is given by Equation (1)}$$

Note the right-hand-side integral was defined in the previous section.

**Definition. Regions of Type 1, 2, and 3. Elementary Regions**

A region of type $1$ is a subset $D$ of $\mathbb{R}^2$ of the form

$$D = \{(x, y) \mid a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x)\},$$

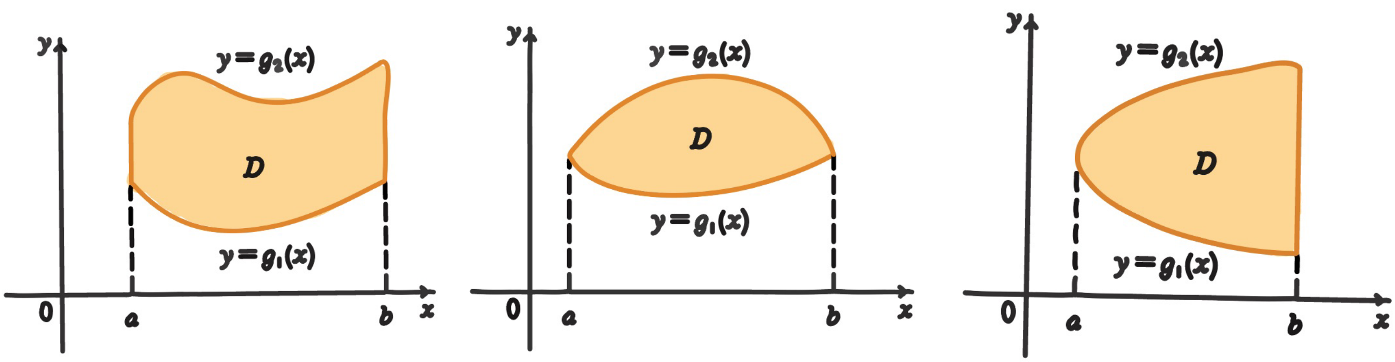
where $g_1$ and $g_2$ are continuous on $[a, b]$.

A region of type 2 is defined by

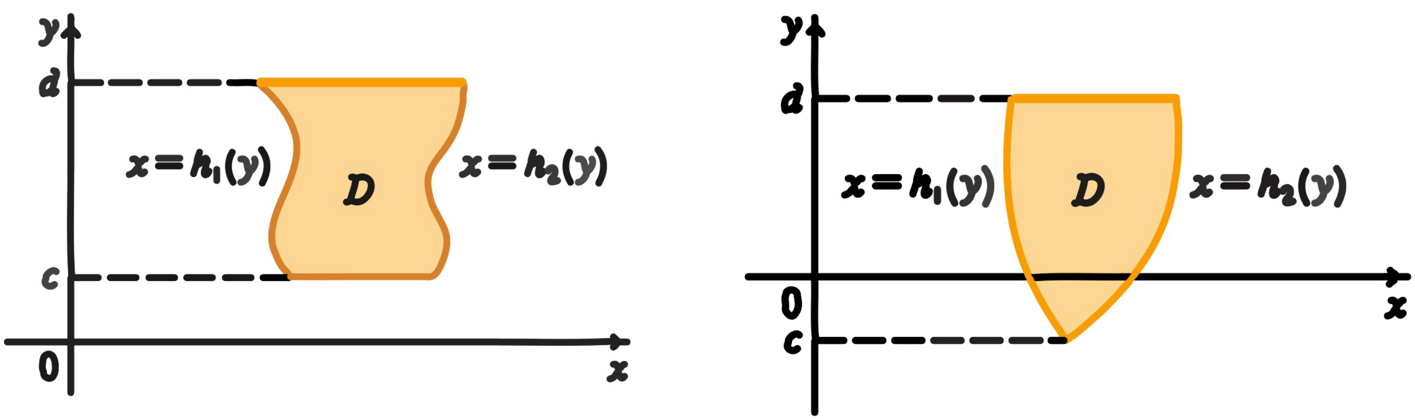
$$D = \{(x, y) \mid c \leqslant y \leqslant d, h_1(y) \leqslant x \leqslant h_2(y)\},$$

where $h_1$ and $h_2$ are continuous.

We say that $D$ is a region of type 3 if it is of both type 1 and type 2. A region of type 1,2 , or 3 is called an elementary region.




Regions of type1




Regions of type 2

To evaluate $\iint_D f(x,y)dA,$ when $D$ is a region of type $1$, we choose a rectangle $R=[a,b]\times[c,d]$ that contains $D$.

Let $F$ be the function given by Equation 1. Then, by Fubini's Theorem,

$$\iint_D f(x,y)dA=\iint_R F(x,y)dA=\int_a^b\int_c^d F(x,y)dydx$$

Observe that $F(x,y)=0$ if $y<g_1(x)$ or $y>g_2(x)$ because $(x,y)$ then lies outside $D$.

Therefore

$$\int_c^d F(x,y)dy=\int_{g_1(x)}^{g_2(x)} F(x,y)dy=\int_{g_1(x)}^{g_2(x)} f(x,y)dy$$

because $F(x,y)=f(x,y)$ when $g_1(x)\leqslant y\leqslant g_2(x)$.

Therefore we have the following formula that enables us to evaluate the double integral as an iterated integral.

If $f$ is continuous on a type $1$ region $D$ such that

$$D=\{(x,y)\mid a\leqslant x\leqslant b, g_1(x)\leqslant y\leqslant g_2(x)\}$$

then

$$\iint_D f(x,y)dA=\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

Similarly, we have

If $f$ is continuous on a type $2$ region $D$ such that

$$D=\{(x,y)\mid c\leqslant y\leqslant d, h_1(y)\leqslant x\leqslant h_2(y)\}$$

then

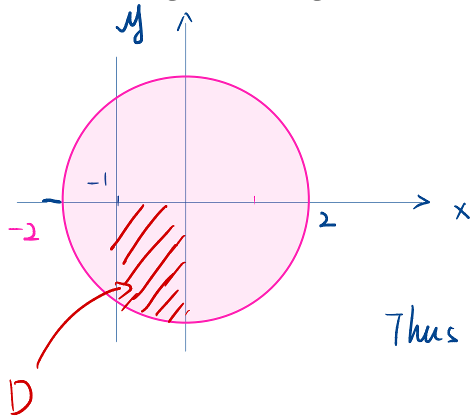
$$\iint_D f(x,y)dA=\int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

**Example 1.**

For the integral

$$\int_{-1}^{0}\int_{-\sqrt{4-x^2}}^{0} xy\,dy\,dx$$

sketch the region of integration and evaluate the integral.




ANS: Note $-\sqrt{4-x^2} \le y \le 0$

When $y = -\sqrt{4-x^2} \Rightarrow y^2 = 4 - x^2$

$\Rightarrow x^2 + y^2 = 4$ circle centered at

(0,0) with r = 2

Thus

$$\int_{-1}^{0}\int_{-\sqrt{4-x^2}}^{0} xy\,dy\,dx$$

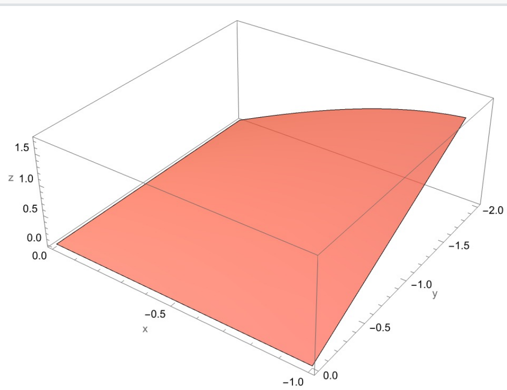
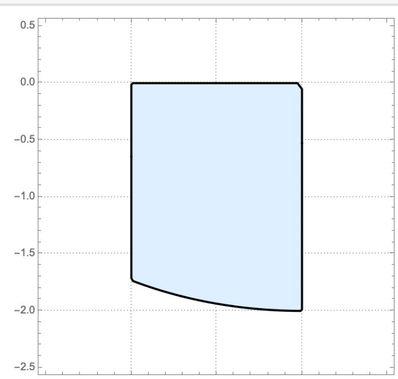
$$= \int_{-1}^{0} \frac{1}{2} x y^2 \Big|_{-\sqrt{4-x^2}}^{0} dx$$

$$= \int_{-1}^{0} \underbrace{\left[0 - \frac{1}{2} x \cdot \left(-\sqrt{4-x^2}\right)^2\right]}_{A(x)} dx$$

$$= -\frac{1}{2}\int_{-1}^{0} (4x - x^3)\,dx = -\frac{1}{2} \cdot \left[2x^2 - \frac{1}{4}x^4\right]\Big|_{-1}^{0}$$

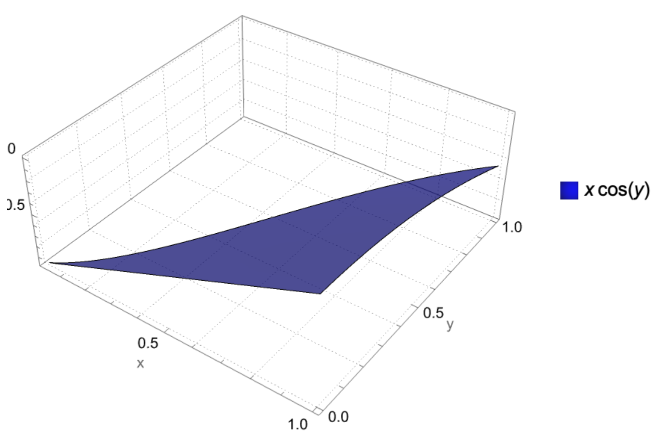
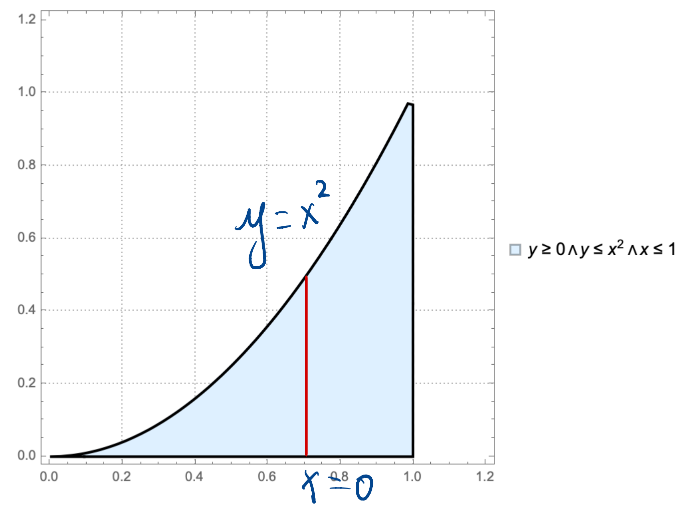
$$= -\frac{1}{2}\left[0 - 2\cdot(-1)^2 - \frac{1}{4}\left(0 - (-1)^4\right)\right] = -\frac{1}{2}\left[-2 + \frac{1}{4}\right]$$

$$= 1 - \frac{1}{8} = \frac{7}{8}$$

| Graph of $f(x)$ | Region $D$ |
|---|---|
|  |  |

**Example 2.**

Evaluate the double integral $\iint_D x\cos y\, dA$, where $D$ is bounded by $y = 0, y = x^2$, and $x = 1$.

| **Graph of $f(x) = x\cos y$** | **Region $D$** |
|---|---|
|   |   |

ANS: We start from drawing the region bounded by $y=0$, $y=x^2$, $x=1$.

We have $\iint_D x\cos y\, dA = \int_0^1 x \int_0^{x^2} \cos y\, dy\, dx$ (type 1)

$$= \int_0^1 x \sin y \Big|_{y=0}^{y=x^2} dx = \int_0^1 x \sin x^2 \, dx$$

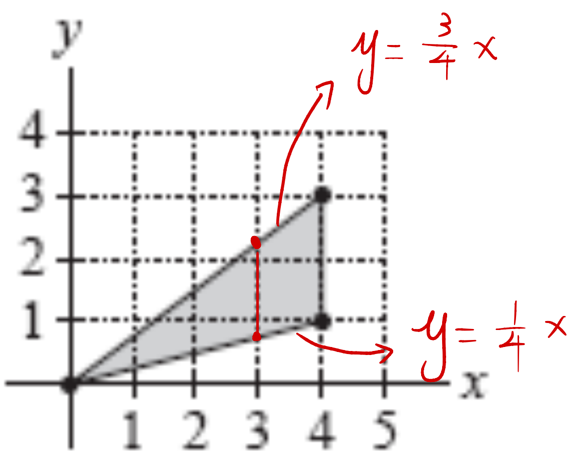
$$= \frac{1}{2}\int_0^1 \sin x^2 \, dx^2 = -\frac{1}{2} \cos x^2 \Big|_0^1$$

$$= -\frac{1}{2}\left(\cos 1^2 - \cos 0\right)$$

$$= \frac{1}{2}\left(1 - \cos 1\right).$$

**Example 3.**

Calculate the double integral of $f(x, y) = -8ye^x$ over the triangle indicated in the following figure:




Notice the triangle region can be expressed by

$$0 \le x \le 4$$

$$\frac{1}{4}x \le y \le \frac{3}{4}x$$

Thus $\iint_D -8ye^x \, dA = \int_0^4 e^x \int_{\frac{1}{4}x}^{\frac{3}{4}x} -8y \, dy \, dx = \int_0^4 e^x \left(-4y^2\right)\Big|_{\frac{1}{4}x}^{\frac{3}{4}x} dx$

$$= \int_0^4 e^x \left[-4\left(\frac{9}{16}x^2 - \frac{1}{16}x^2\right)\right] dx$$

$$= \int_0^4 e^x \cdot (-2x^2)\, dx = -2\int_0^4 x^2 e^x \, dx$$

We check the table of integral.

**96.** $\int ue^{au}\, du = \frac{1}{a^2}(au - 1)e^{au} + C$

**97.** $\int u^n e^{au}\, du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\, du$

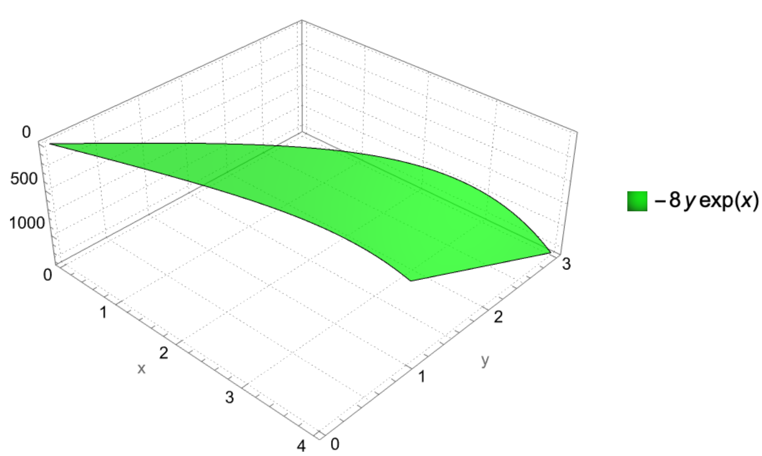
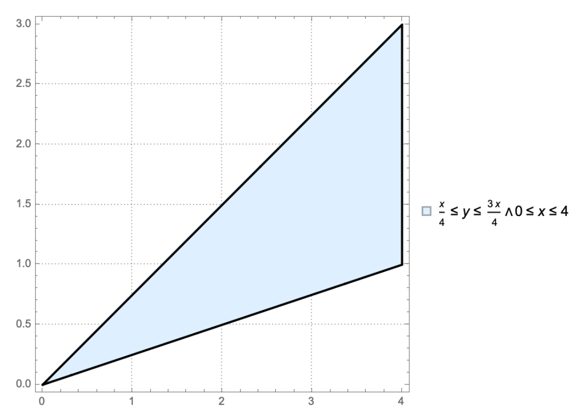
Then $\underbrace{\int x^2 e^x \, dx}_{\text{using 97, with } x = u,\ n = 2,\ a = 1} = x^2e^x - \underbrace{2\int xe^x dx}_{\text{using 96 with } x = u,\ a = 1} = x^2e^x - 2\left[(x-1)e^x\right] + C$

$$= x^2e^x - 2xe^x + 2e^x + C.$$

$$-2\int_0^4 x^2e^x\, dx = -2\left[x^2e^x - 2xe^x + 2e^x\right]\Big|_0^4$$

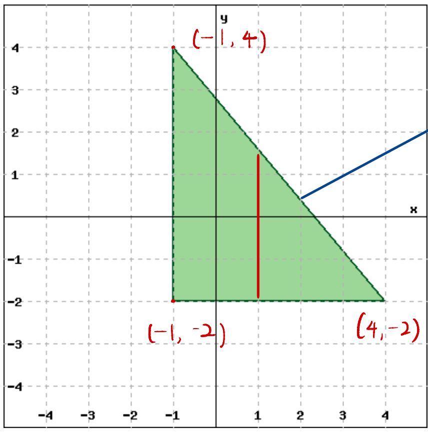
$$= -2\left[4^2e^4 - 8e^4 + 2e^4 - (0 - 0 + 2e^0)\right]$$

$$= -20e^4 + 4$$

| Graph of $f(x) = -8ye^x$ | Region $D$ |
| --- | --- |
|   |   |

**Exercise 4.**

Suppose $R$ is the shaded region in the figure, and $f(x,y)$ is a continuous function on $R$. Find the limits of integration for the following iterated integrals.




$$y+2=\frac{4-(-2)}{-1-4}(x-4)=-\frac{6}{5}(x-4)$$

$$\Rightarrow y=-\frac{6}{5}(x-4)-2 \quad (y \text{ in terms of } x)$$

Also $x-4=-\frac{5}{6}(y+2)$

$$\Rightarrow x=-\frac{5}{6}(y+2)+4 \quad (x \text{ in terms of } y)$$

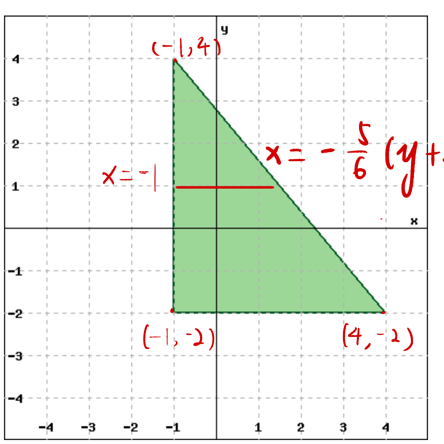
(a) $\iint_R f(x,y)dA=\int_A^B\int_C^D f(x,y)\,dy\,dx$ (type 1)

The region is bounded by $y=-2$ and $y=-\frac{6}{5}(x-4)-2$.

and $-1\leq x\leq 4$

Thus $\iint_R f(x,y)dA=\int_A^B\int_C^D f(x,y)\,dy\,dx=\int_{-1}^{4}\int_{-2}^{-\frac{6}{5}(x-4)-2} f(x,y)\,dy\,dx$

(b) $\iint_R f(x,y)dA=\int_E^F\int_G^H f(x,y)\,dx\,dy$ (type 2)




The region is bounded on the left by $x=-1$ and on the right by $x=-\frac{5}{6}(y+2)+4$.

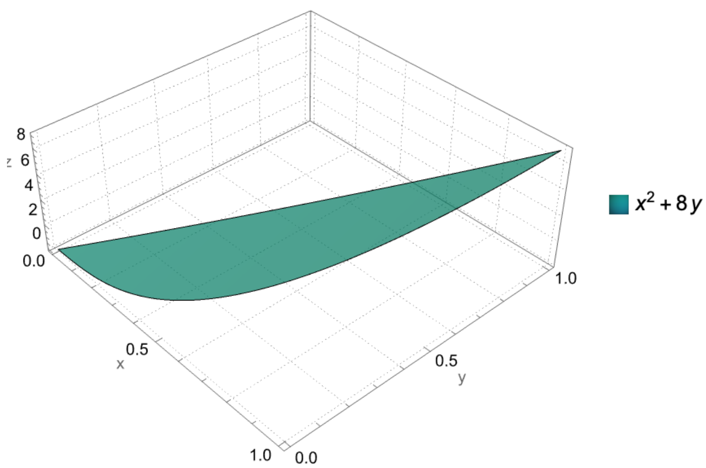
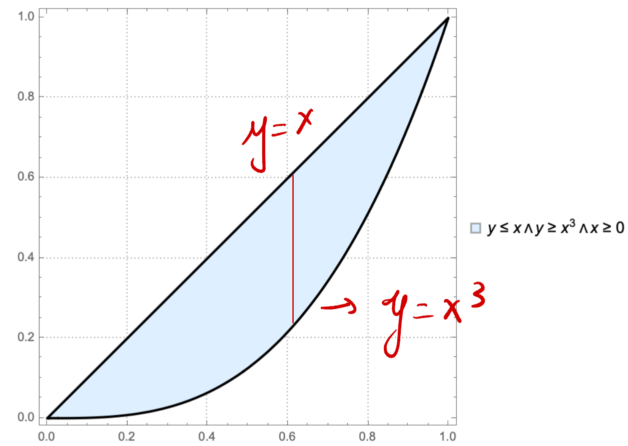
The bounds for $y$ are from $-2$ to $1$.

Thus

$$\int_E^F\int_G^H f(x,y)\,dx\,dy=\int_{-2}^{1}\int_{-1}^{-\frac{5}{6}(y+2)+4} f(x,y)\,dx\,dy$$

**Exercise 5.**

Evaluate the double integral $\iint_D (x^2 + 8y)\, dA$, where $D$ is bounded by $y = x$, $y = x^3$, and $x \geq 0$

| Graph of $f(x) = x^2 + 8y$ | Region $D$ |
| --- | --- |
|   |   |

ANS: We start from drawing the region bounded by

$y = x$, $y = x^3$, $x \geq 0$.

Thus we evaluate

$$\iint_D (x^2+8y)\, dA = \int_0^1 \int_{x^3}^{x} x^2 + 8y \; dy\, dx$$

$$= \int_0^1 \left[x^2 y + 4y^2\right]\Big|_{x^3}^{x} dx = \int_0^1 \left[x^2(x - x^3) + 4(x^2 - x^6)\right] dx$$

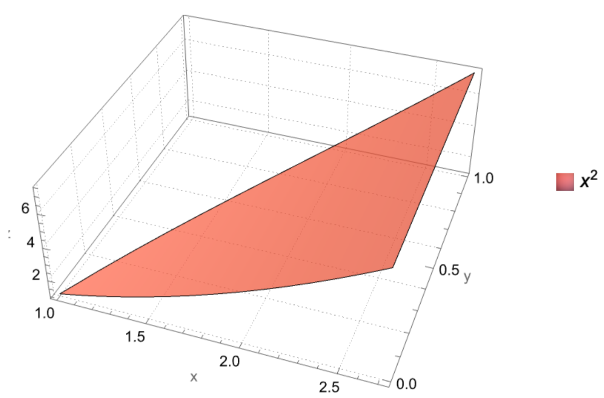
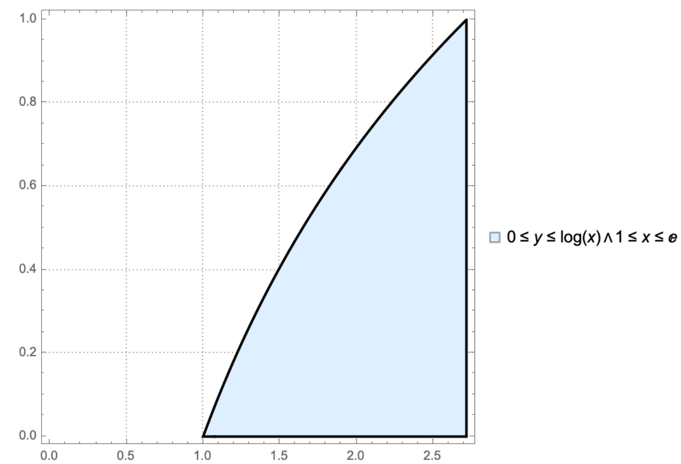
$$= \int_0^1 \left(x^3 - x^5 + 4x^2 - 4x^6\right) dx$$

$$= \left[\frac{1}{4}x^4 - \frac{1}{6}x^6 + \frac{4}{3}x^3 - \frac{4}{7}x^7\right]\Big|_0^1$$

$$= \frac{1}{4} - \frac{1}{6} + \frac{4}{3} - \frac{4}{7} = \frac{71}{84}$$

**Exercise 6.**

Evaluate the double integral $\iint_D x^2 dA$, where $D = \{(x, y) : 1 \le x \le e, 0 \le y \le \ln x\}$

<table>
<tr><th>Graph of $f(x)$</th><th>Region $D$</th></tr>
<tr><td>
</td><td>
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ANS: $\iint_D x^2 dA = \int_1^e \int_0^{\ln x} x^2 \, dy \, dx$

$= \int_1^e x^2 y \Big|_0^{\ln x} dx$

$= \int_1^e x^2 \ln x \, dx$

To compute the antiderivative $\int x^2 \ln x \, dx$, we use integration by parts:

$$\int u \, dv = uv - \int v \, du$$

Rewrite $\int x^2 \ln x \, dx = \frac{1}{3} \int \ln x \, dx^3$

Let $u = \ln x$, $v = x^3$, then

$$\frac{1}{3}\int \ln x \, dx^3 = \frac{1}{3}\left[x^3 \ln x - \int x^3 \cancelto{\frac{1}{x}dx}{d \ln x}\right]$$

$$= \frac{1}{3}\left[x^3 \ln x - \int x^2 \, dx\right] = \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3.$$

Thus

$$\int_1^e x^2 \ln x \, dx$$

$$= \left[\frac{1}{3} x^3 \ln x - \frac{1}{9} x^3\right]\Bigg|_1^e$$

$$= \frac{1}{3} e^3 \cancelto{1}{\ln e} - \frac{1}{9} e^3 - \left(\frac{1}{3} \cdot 1^3 \cdot \cancelto{0}{\ln 1} - \frac{1}{9} \cdot 1^3\right)$$

$$= \frac{2}{9} e^3 + \frac{1}{9}$$

$$= \frac{1}{9}(1 + 2e^3)$$